Math 105 Chapter 11: Convergence Tests.

Suppose we have a series

$$\sum_{n=1}^{\infty} a_n$$

I want to determine if it converges or not. Going through the process of finding the partial sums and taking limits can get very complicated, especially when a_n is complicated.

eg Suppose I want to know if

$$\sum_{n=1}^{\infty} \frac{(2n)!\,\pi^n}{n^5\, n^n}$$

converges or not. Working with partial sums is too complicated. Our goal is to create tests that can help us determin if a series converges or not.

First lets suppose $\sum_{n=1}^{\infty} a_n$ converges to L. What does that tell us about $\lim_{n\to\infty} a_n$?

Well
$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k$$

$$= S_n - S_{n-1}$$

Where $S_n = \sum_{k=1}^{n} a_k$ is the k^{th} partial sum.

So $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$
$$= L - L$$
$$= 0$$

So $\lim_{n\to\infty} a_n = 0$

Thus we have our first test for divergence.

Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

eg Does $\sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2+4}}$ converge or diverge?

sol^n $$\lim_{n\to\infty} \frac{n}{\sqrt{3n^2+4}} = \lim_{n\to\infty} \frac{n}{\sqrt{3n^2+4}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$
$$= \lim_{n\to\infty} \frac{1}{\sqrt{3+\frac{4}{n^2}}}$$
$$= \frac{1}{\sqrt{3}}$$
$$\neq 0$$

Thus the series diverges by the divergence test.

Eg Does $\sum_{n=1}^{\infty} \cos n$ converge or diverge?

Since $\lim_{n\to\infty} \cos n$ does not exist, we have the series diverges.

Eg Does $\sum_{n=1}^{\infty} \log\left(\frac{1}{n}\right)$ converge or diverge?

$$\lim_{n\to\infty} \log\left(\frac{1}{n}\right) = -\lim_{n\to\infty} \log n = -\infty$$

Remarks:

- The divergergence test should be the first test you should try, since the others in general require more work.
- Just because $\lim_{n\to\infty} a_n = 0$ does <u>not</u> mean $\sum_{n=1}^{\infty} a_n$ converges. I just means you have to do more work.

eg $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series. Divergence test fails since

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

But!

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

$$\geq 1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{1} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{6} + \frac{1}{6}}_{\frac{1}{3}} + \underbrace{\frac{1}{8} + \frac{1}{8}}_{\frac{1}{4}} + \cdots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n}$$

So we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \sum_{n=1}^{\infty} \frac{1}{n}$$

Which is clearly false unless $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Thus $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

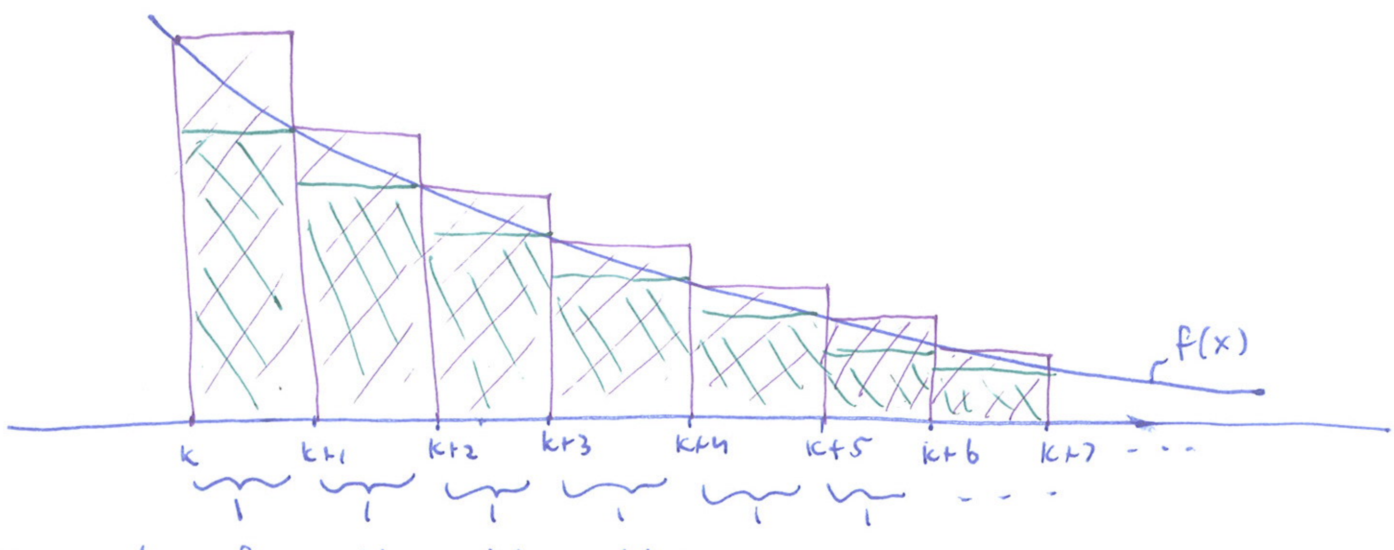
The next test will give an alternate proof of this.

<u>Theorem</u>: (Integral Test)

Suppose $f(x)$ is a non negative and non-increasing for $n \geq k$.

- If $\int_k^{\infty} f(x)\,dx$ converges then $\sum_{n=k}^{\infty} f(n)$ converges.
- If $\int_k^{\infty} f(x)\,dx$ diverges then $\sum_{n=k}^{\infty} f(n)$ diverges.

Just to give an idea of why this is true



It is clear from the picture that

$$\text{green area} \leq \int_k^{\infty} f(x)dx \leq \text{purple area}$$

purple area is

$$f(k) + f(k+1) + f(k+2) \cdots$$

$$= \sum_{n=k}^{\infty} f(n)$$

green area is

$$f(k+1) + f(k+2) + f(k+3) + \cdots$$

$$= \sum_{n=k+1}^{\infty} f(n)$$

So if $\int_k^\infty f(x)\,dx$ is finite then green area is finite, since

$$\sum_{n=k+1}^{\infty} f(k) \leq \int_k^\infty f(x)\,dx$$

If $\int_k^\infty f(x)\,dx$ is infinite then purple area is infinite since

$$\int_k^\infty f(x)\,dx \leq \sum_{n=k}^{\infty} f(n)$$

A special case of this is the p-series test:

<u>Theorem</u>: (P-Test) <u>Very Important</u>

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l}\text{converges if } p > 1 \\ \text{diverges if } p \leq 1\end{array}$$

<u>proof</u>: When $p<0$, we have $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.

$p=0$, we have $\lim_{n\to\infty} \frac{1}{n^p} = 1$

So by divergence test, the series diverges.

when $p>0$ then $\frac{1}{x^p}$ is decreasing when $x \geq 1$. So by integral test, Our series converges whenever $\int_1^\infty \frac{1}{x^p}\,dx$ does. We showed before that $\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p>1$ and diverges otherwise.

eg Does $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converge or diverge?

$f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$ is non-negative and non-decreasing when $x \geq 1$, so

$$\int_1^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx, \quad u=\sqrt{x}, \quad du = \frac{1}{2\sqrt{x}}\,dx$$

$$= \lim_{t\to\infty} \int_1^{\sqrt{t}} 2e^{-u}\,du$$

$$= \lim_{t\to\infty} -2e^{-u}\Big|_1^{\sqrt{t}}$$

$$= -2 \lim_{t\to\infty} \left[e^{-\sqrt{t}} - e^{-1}\right]$$

$$= \frac{2}{e}$$

So the integral converges. Thus by the integral test the series converges.

Note we don't need our function to be always decreasing, it just need to be decreasing <u>eventually</u>

eg. Does $\sum_{n=1}^{\infty} \frac{\log n}{n}$ converge or diverge?

Let $f(x) = \frac{\log x}{x}$, $x \geq 1$. Since $\log x \geq 0$ when $x \geq 1$

we have $f(x) \geq 0$ when $x \geq 1$.

$$f'(x) = \frac{\frac{1}{x} \cdot x - \log x}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

< 0 when $x > e$

So when $x > e$ $f(x)$ is decreasing. So we can apply integral test.

$$\int_1^\infty \frac{\log x}{x}\, dx = \lim_{t \to \infty} \int_1^t \frac{\log x}{x}\, dx, \quad u = \log x, \quad du = \frac{1}{x}\, dx$$

$$= \lim_{t \to \infty} \int_0^{\log t} u\, du$$

$$= \lim_{t \to \infty} \frac{u^2}{2} \Big|_0^{\log t}$$

$$= \lim_{t \to \infty} (\log t)^2$$

$$= \infty$$

Thus by integral test $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges.

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Let us look at our last example again. When $n \geq 3$, we have $\log n \geq 1$. So

$$\frac{\log n}{n} \geq \frac{1}{n}$$

It should follow that

$$\sum_{n=3}^{\infty} \frac{\log n}{n} \geq \sum_{n=3}^{\infty} \frac{1}{n}$$

The left hand side is ∞ by the p-test. So we can.. say that

$$\sum_{n} \frac{\log n}{n} = \infty$$

This is a strategy that very useful. We have an ugly series that we don't know if it converges or not. But can determine it's convergence by comparing it with a nice series whose convergence we are sure of. This idea is sumarized in the next theorem:

<u>Theorem</u>: (Comparison Test) Let $a_n, b_n \geq 0$ be sequences.

- If $a_n \leq b_n$ (for n large enough), and $\sum_{n=1}^{\infty} b_n$ converges, then
$$\sum_{n=1}^{\infty} a_n \text{ converges.}$$
- If $b_n \leq a_n$ (for n large enough), and $\sum_{n=1}^{\infty} b_n$ diverges, then
$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

eg $\displaystyle\sum_{n=6}^{\infty} \frac{1}{\sqrt{n-5}}$

Well note $n-5 \leq n$ so

$$\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n-6}}$$

Since $\displaystyle\sum_{n=6}^{\infty} \frac{1}{\sqrt{n}}$ diverges, $\displaystyle\sum_{n\geq 6}^{\infty} \frac{1}{\sqrt{n-5}}$ diverges by comparison test.

eg $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n.}{n^3}$

we have $0 \leq \dfrac{\sin^2 n}{n^3} \leq \dfrac{1}{n^3}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-test, so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$ converges.

eg $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+1}{n^5+n^4+1}$

Since $n^5+n^4+1 > n^5$, we have

$$\frac{n^3+1}{n^5+n^4+1} \leq \frac{n^3+1}{n^5} = \frac{1}{n^2} + \frac{1}{n^5}$$

Since $\displaystyle\sum_n \frac{1}{n^2}$, $\displaystyle\sum_n \frac{1}{n^5}$ converge by p-test, so by comparison test,
$\displaystyle\sum_{n=1}^{\infty} \frac{n^3+1}{n^5+n^4+1}$ converges.

Theorem: (Limit Comparison Test)

Let $a_n \geq 0$ and $b_n > 0$ eventually for large enough n.

Further suppose
$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where $0 < L < \infty$, then

- If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

eg $\sum_{n=1}^{\infty} \frac{5n-1}{2n^3-3n^2+10}$, let $a_n = \frac{5n-1}{2n^3-3n^2+10}$

we want to find a sequene b_n to compare a_n to.

Well when n is large

$$5n-1 \approx 5n$$
$$2n^3-3n^2+10 \approx 2n^3$$

So $$a_n = \frac{5n-1}{2n^3-3n^2+10} \approx \frac{5n}{2n^3} = \frac{5}{2n^2}$$

So let $b_n = \frac{1}{n^2}$, and lets see what happens.

$$\frac{a_n}{b_n} = \frac{\frac{5n-1}{2n^3-3n^2+10}}{\frac{1}{n^2}} = \frac{5n^3-n^2}{2n^3-3n^2+10}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n^3 - n^2}{2n^3 - 3n^2 + 10}$$

$$= \lim_{n\to\infty} \frac{5n^3 - n^2}{2n^3 - 3n^2 + 10} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{5 - \frac{1}{n}}{2 - \frac{3}{n} + \frac{10}{n^3}}$$

$$= \frac{5}{2}$$

Since $0 < \frac{5}{2} < \infty$, we have that by limit comparison test

$\sum_{n=1}^{\infty} \frac{5n-1}{2n^3-3n^2+10}$ converges or diverges if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test so $\sum_{n=1}^{\infty} \frac{5n-1}{2n^3-3n^2+10}$ converges.

Fun Fact! $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, FUN!

eg. $\sum_{n=2}^{\infty} \frac{2n\sqrt{n+1}}{(8n^6-9)^{1/3}}$, let $a_n = \frac{2n\sqrt{n+1}}{(8n^6-9)^{1/3}}$

We lets see what a_n looks like when n is large.

For large n, $n\sqrt{n+1} \approx n\sqrt{n}$, since $n+1 \approx n$.

so $2n\sqrt{n+1} \approx 2n^{3/2}$

Similarly $(8n^6 - 9)^{1/3} \approx (8n^6)^{1/3}$, since $n^6 - 9 \approx n^6$
$= 2n^2$

So $a_n = \dfrac{2n\sqrt{n+1}}{(8n^6-9)^{1/3}}$

$\approx \dfrac{n^{3/2}}{2n^2}$

$= \dfrac{1}{2\sqrt{n}}$

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-test ($p = \frac{1}{2}$), we want to say the series diverges. Let us prove it by limit comparison test.

Let $b_n = \frac{1}{\sqrt{n}}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{n\sqrt{n+1}}{(8n^6-9)^{1/3}}}{\dfrac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n+1}}{(8n^6-9)^{1/3}}$$ ← highest order is $\frac{3}{2} + \frac{1}{2} = 2$ ← highest order is $\frac{6}{3} = 2$

$$= \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n+1}\ \frac{1}{n^2}}{(8n^6-9)^{1/3}\ \frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{\dfrac{n+1}{n}}}{\left(\dfrac{8n^6-9}{n^6}\right)^{1/3}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{\left(8-\frac{9}{n^6}\right)^{1/3}}$$

$$= \frac{\sqrt{1+0}}{(8-0)^{1/3}}$$

$$= \frac{1}{2}$$

Thus $\sum_{n=2}^{\infty} a_n$ diverges since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges by limit comparison test.

eg $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$, $a_n = \frac{1}{2^n-1}$

When n is large $2^n - 1 \approx 2^n$ so

$$\frac{1}{2^n-1} \approx \frac{1}{2^n}$$

Let $b_n = \frac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n-1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n-1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}}$$

$$= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}}$$

$$= \lim_{n \to \infty} \frac{1}{1-0}$$

$$= 1$$

So by limit comparison test $\sum_{n=1}^{\infty} a_n$ converges since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with ratio $r = \frac{1}{2} < 1$, and thus converges.

Absolute Convergence / Ratio test.

So far we have only built strategies to determine convergence of series with positive terms. Eg. Integral test, comparison, limit comparison test all require you to only look at positive sums. Things get a bit wonky when you allow negative terms into the mix.

Theorem: If $\sum_{n\geq 1}^{\infty} |a_n|$ converges, then so does $\sum_{n\geq 1}^{\infty} a_n$.

proof: Since $$-|a_n| \leq a_n \leq |a_n|$$

$$\Rightarrow \quad 0 \leq |a_n| + a_n \leq 2|a_n|$$

So by comparison test, $\sum_{n\geq 1}^{\infty} |a_n| + a_n$ converges since $2\sum_{n\geq 1}^{\infty} |a_n|$ does.

Let $L = \sum_{n\geq 1}^{\infty} |a_n| + |a_n|$.

$\Rightarrow \sum_{n\geq 1}^{\infty} a_n = L - \sum_{n=1}^{\infty} |a_n|$, which converges.

eg $\sum_{n\geq 1}^{\infty} \frac{\sin n}{2^n}$,

All our test are useless since $\sin n$ is not neccerarilly positive.

Let us show $\sum_{n\geq 1}^{\infty} \left| \frac{\sin n}{2^n} \right|$ converges.

Now $\left| \frac{\sin n}{2^n} \right| \leq \frac{1}{2^n}$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, by comparison test

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n} \right| \text{ converges.}$$

By the theorem $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges.

The question now becomes, does the opposite hold? Ie. If we have $\sum_n a_n$ converges, then does $\sum_n |a_n|$ converge? The answer is no!

eg $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

We have already seen that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (p-test, $p=1$).

Once can show using methods that you are not responsible for (look up alternating series test if interested), that the above series converges. We will actually show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2 \quad (\text{WOW!})$$

Now let us give a name to this phenomenon.

<u>Definition</u>: We say $\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

<u>converges conditionally</u> if $\sum_{n=1}^{\infty} |a_n|$ diverges.

Now we ask, when does a series converge absolutely? There are many ways. We will only talk about one, the ratio test.

<u>Theorem</u>: (Ratio test)

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

<u>Remarks</u>:

① If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ or doesn't exist the test is inconclusive.

② $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ means for large n,

$$|a_{n+1}| \approx L|a_n|$$

- So when $L < 1$, we have

$$|a_{n+1}| < L|a_n| < L^2|a_{n-1}| \cdots < L^n|a_1|$$

So $\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} L^{n-1}|a_1| = |a_1| \underbrace{\sum_{n=1}^{\infty} L^{n-1}}_{\text{geometric series.}} < \infty$

So $\sum_{n=1}^{\infty} |a_n|$ converges absolutely.

• $L > 1$ means that

$$|a_{n+1}| \approx L\,|a_n| > |a_n|$$

So $|a_n|$ is an increasing sequence and does not go to zero, thus diverges.

Let us see how to use this test.

eg $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(-5)^n}{n!}}_{a_n}$ $\left[\text{Recall: } n! = \underbrace{1 \cdot 2 \cdot 3 \cdots (n-1)}_{(n-1)!}\, n = n\,(n-1)!\right]$

$$\lim_{n\to\infty} \left| a_{n+1}/a_n \right| = \lim_{n\to\infty} \left| \frac{(-5)^{n+1}}{(n+1)!} \Big/ \frac{(-5)^n}{n!} \right|$$

$$= \lim_{n\to\infty} \frac{5^{n+1}\, n!}{5^n\,(n+1)!}$$

$$= \lim_{n\to\infty} \frac{5\, n!}{(n+1)\, n!}$$

$$= \lim_{n\to\infty} \frac{5}{n+1}$$

$$= 0$$

$$< 1$$

So by ratio test, the series converges.

Fun Fact: $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{n!} = e^{-5}$

eg $\sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{3^n}{n^{50}}}_{a_n}$

$$\lim_{n\to\infty} \left| a_{n+1}/a_n \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 3^{n+1}}{(n+1)^{50}} \Big/ \frac{(-1)^n 3^n}{n^{50}} \right|$$

$$= \lim_{n\to\infty} \frac{3^{n+1} n^{50}}{3^n (n+1)^{50}}$$

$$= \lim_{n\to\infty} 3 \left(\frac{n}{n+1}\right)^{50}$$

$$= 3 \left(\lim_{n\to\infty} \frac{n}{n+1} \right)^{50}$$

$$= 3 \left(\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \right)^{50}$$

$$= 3 \cdot \left(\frac{1}{1+0} \right)^{50}$$

$$= 3$$

$$> 1$$

So $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n^{50}}$ diverges.

eg $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$, $a_n = \frac{(-2)^n}{n^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(n+1)^{n+1}} \Big/ \frac{(-2)^n}{n^n} \right|$$

$$= \lim_{n\to\infty} \frac{2^{n+1} n^n}{2^n (n+1)^{n+1}}$$

$$= 2 \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}}$$

So we need to find $\lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}}$. Well notice

$$0 \le \frac{n^n}{(n+1)^{n+1}}$$

$$= \frac{n^n}{(n+1)(n+1)^n}$$

$$= \frac{1}{n+1} \left(\frac{n}{n+1}\right)^n$$

$$\le \frac{1}{n+1}, \quad \text{since } \frac{n}{n+1} \le 1$$

Thus $0 \le \frac{n^n}{(n+1)^{n+1}} \le \frac{1}{n+1}$

By squeeze theorem, $\lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = 0$

Thus $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ converges absolutely.

Now let us show that this test is not perfect. When $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ this tells us nothing.

eg $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{1}{n+1} \Big/ \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= 1$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-test ; $p=1$)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{1}{(n+1)^2} \Big/ \frac{1}{n^2}$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= 1$$

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-test, $p=2$),

FUN Fact! It actually is $\frac{\pi^2}{6}$

So for both $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, we have

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.